ON MANY-SORTED ALGEBRAIC CLOSURE OPERATORS

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Abstract. A theorem of Birkhoff-Frink asserts that every algebraic closure operator on an ordinary set arises, from some algebraic structure on the set, as the operator that constructs the subalgebra generated by a subset. However, for many-sorted sets, i.e., indexed families of sets, such a theorem is not longer true without qualification. We characterize the corresponding many-sorted closure operators as precisely the uniform algebraic operators.

Some theorems of ordinary universal algebra can not be automatically generalized to many-sorted universal algebra, e.g., Matthiessen [5] proves that there exist many-sorted algebraic closure systems that can not be concretely represented as the set of subalgebras of a many-sorted algebra. As is well known, according to a representation theorem of Birkhoff and Frink [1], this is not so for the single-sorted algebraic closure systems.

In [2] it was obtained a concrete representation for the so-called many-sorted uniform 2-algebraic closure operators. However, as will be proved below, confirming a conjecture by A. Blass in his review of [2], the main result in [2] remains true if we delete from the above class of many-sorted operators the condition of 2-algebraicity. Therefore a many-sorted algebraic closure operator will be concretely representable as the set of subalgebras of a many-sorted algebra iff it is uniform. We point out that the proof we offer follows substantially that in Grätzer [4] for the single-sorted case, but differs from it, among others things, by the use we have to make, on the one hand, of the concept of uniformity, missing in the single-sorted case, and, on the other hand, of the Axiom of Choice, because of the lack, in the many-sorted case, of a canonical choice in the definition of the many-sorted operations.

In what follows we use, for a set of sorts S and an S-sorted signature Σ, the concept of many-sorted Σ-algebra and subalgebra in the standard meaning, see e.g., [3].

To begin with, as for ordinary algebras, also the set of subalgebras of a many-sorted algebra is an algebraic closure system.

Proposition 1. *Let $\underline{A}$ be a many-sorted Σ-algebra. Then the set of all subalgebras of $\underline{A}$, denoted by $\mathrm{Sub}(\underline{A})$, is an algebraic closure system on A, i.e., we have*

1. $A \in \mathrm{Sub}(\underline{A})$.
2. *If I is not empty and $(X^i)_{i\in I}$ is a family in $\mathrm{Sub}(\underline{A})$, then $\bigcap_{i\in I} X^i$ is also in $\mathrm{Sub}(\underline{A})$.*
3. *If I is not empty and $(X^i)_{i\in I}$ is an upwards directed family in $\mathrm{Sub}(\underline{A})$, then $\bigcup_{i\in I} X^i$ is also in $\mathrm{Sub}(\underline{A})$.*

However, as we will prove later on, in the many-sorted case the many-sorted algebraic closure operator canonically associated to the algebraic closure system of the subalgebras of a many-sorted algebra has an additional and characteristic property, that of being uniform.

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Now we recall the concept of support of a sorted set and that of many-sorted algebraic closure operator on a sorted set, essentials to define that of many-sorted algebraic uniform closure operator.

Definition 1. Let A be an S-sorted set. Then the *support* of A, denoted by $\mathrm{supp}(A)$, is the subset $\{\, s \in S \mid A_s \neq \varnothing \,\}$ of S.

Definition 2. Let A be an S-sorted set. A *many-sorted algebraic closure operator* on A is an operator J on $\mathrm{Sub}(A)$, the set of all S-sorted subsets of A, such that, for every $X, Y \subseteq A$, satisfies:

1. $X \subseteq J(X)$, i.e., J is extensive.
2. If $X \subseteq Y$, then $J(X) \subseteq J(Y)$, i.e., J is isotone.
3. $J(J(X)) = J(X)$, i.e., J is idempotent.
4. $J(X) = \bigcup_{F \in \mathrm{Sub_f}(X)} J(F)$, i.e., J is algebraic, where a part F of X is in $\mathrm{Sub_f}(X)$, the set of *finite S-sorted subsets* of X, iff $\mathrm{supp}(F)$ is finite and, for every $s \in \mathrm{supp}(F)$, F_s is finite.

A many-sorted algebraic closure operator J on A is *uniform* iff, for $X, Y \subseteq A$, from $\mathrm{supp}(X) = \mathrm{supp}(Y)$, follows that $\mathrm{supp}(J(X)) = \mathrm{supp}(J(Y))$.

Definition 3. Let $\underline{A}$ be a many-sorted Σ-algebra. We denote by $\mathrm{Sg}_{\underline{A}}$ the many-sorted algebraic closure operator on A canonically associated to the algebraic closure system $\mathrm{Sub}(\underline{A})$. If $X \subseteq A$, $\mathrm{Sg}_{\underline{A}}(X)$ is the *subalgebra of $\underline{A}$ generated by X*.

Next, as for ordinary algebras, we define for a many-sorted Σ-algebra $\underline{A}$ an operator on $\mathrm{Sub}(A)$ that will allow us to obtain, for every subset of A, by recursion, an $\mathbb{N}$-ascending chain of subsets of A from which, taking the union, we will obtain an equivalent, but more constructive, description of the subalgebra of $\underline{A}$ generated by a subset of A. Moreover, we will make use of this alternative description to prove the uniformity of the operator $\mathrm{Sg}_{\underline{A}}$ and also in the proof of the representation theorem.

Definition 4. Let $\underline{A} = (A, F)$ be a many-sorted Σ-algebra.

1. We denote by $\mathrm{E}_{\underline{A}}$ the operator on $\mathrm{Sub}(A)$ that assigns to an S-sorted subset X of A, $\mathrm{E}_{\underline{A}}(X) = X \cup \big(\bigcup_{\sigma \in \Sigma_{\cdot,s}} F_\sigma[X_{\mathrm{ar}(\sigma)}] \big)_{s \in S}$, where, for $s \in S$, $\Sigma_{\cdot,s}$ is the set of all many-sorted formal operations σ such that the coarity of σ is s and for $\mathrm{ar}(\sigma) = (s_j)_{j \in m} \in S^\star$, the arity of σ, $X_{\mathrm{ar}(\sigma)} = \prod_{j \in m} X_{s_j}$.
2. If $X \subseteq A$, then the family $(\mathrm{E}^n_{\underline{A}}(X))_{n \in \mathbb{N}}$ in $\mathrm{Sub}(A)$ is such that $\mathrm{E}^0_{\underline{A}}(X) = X$ and $\mathrm{E}^{n+1}_{\underline{A}}(X) = \mathrm{E}_{\underline{A}}(\mathrm{E}^n_{\underline{A}}(X))$, for $n \geq 0$.
3. We denote by $\mathrm{E}^\omega_{\underline{A}}$ the operator on $\mathrm{Sub}(A)$ that assigns to an S-sorted subset X of A, $\mathrm{E}^\omega_{\underline{A}}(X) = \bigcup_{n \in \mathbb{N}} \mathrm{E}^n_{\underline{A}}(X)$

Proposition 2. *Let $\underline{A}$ be a many-sorted Σ-algebra and $X \subseteq A$. Then we have that $\mathrm{Sg}_{\underline{A}}(X) = \mathrm{E}^\omega_{\underline{A}}(X)$.*

Proof. See [2] □

Proposition 3. *Let $\underline{A}$ be a many-sorted Σ-algebra and $X, Y \subseteq A$. Then we have that*

1. *If* $\mathrm{supp}(X) = \mathrm{supp}(Y)$, *then, for every* $n \in \mathbb{N}$, $\mathrm{supp}(\mathrm{E}^n_{\underline{A}}(X)) = \mathrm{supp}(\mathrm{E}^n_{\underline{A}}(Y))$.
2. $\mathrm{supp}(\mathrm{Sg}_{\underline{A}}(X)) = \bigcup_{n \in \mathbb{N}} \mathrm{supp}(\mathrm{E}^n_{\underline{A}}(X))$.
3. *If* $\mathrm{supp}(X) = \mathrm{supp}(Y)$, *then* $\mathrm{supp}(\mathrm{Sg}_{\underline{A}}(X)) = \mathrm{supp}(\mathrm{Sg}_{\underline{A}}(Y))$.

Therefore the many-sorted algebraic closure operator $\mathrm{Sg}_{\underline{A}}$ *is uniform.*

Proof. See [2] □

Finally we prove the representation theorem for the many-sorted uniform algebraic closure operators, i.e., we prove that for an S-sorted set A a many-sorted algebraic closure operator J on $\mathrm{Sub}(A)$ has the form $\mathrm{Sg}_{\underline{A}}$, for some S-sorted signature Σ and some many-sorted Σ-algebra $\underline{A}$ if J is uniform.

Theorem 1. *Let J be a many-sorted algebraic closure operator on an S-sorted set A. If J is uniform, then $J = \mathrm{Sg}_{\underline{A}}$ for some S-sorted signature Σ and some many-sorted Σ-algebra $\underline{A}$.*

Proof. Let $\Sigma = (\Sigma_{w,s})_{(w,s)\in S^{\star}\times S}$ be the S-sorted signature defined, for every $(w,s) \in S^{\star} \times S$, as follows:

$$\Sigma_{w,s} = \{\, (X,b) \in \textstyle\bigcup_{X\in\mathrm{Sub}(A)}(\{X\} \times J(X)_s) \mid \forall t \in S\, (\mathrm{card}(X_t) = |w|_t)\,\},$$

where for a sort $s \in S$ and a word $w\colon |w| \longrightarrow S$ on S, with $|w|$ the lenght of w, the number of occurrences of s in w, denoted by $|w|_s$, is $\mathrm{card}(\{i \in |w| \mid w(i) = s\})$.

We remark that for $(w,s) \in S^{\star} \times S$ and $(X,b) \in \bigcup_{X\in\mathrm{Sub}(A)}(\{X\} \times J(X)_s)$ the following conditions are equivalent:

(1) $(X,b) \in \Sigma_{w,s}$, i.e., for every $t \in S$, $\mathrm{card}(X_t) = |w|_t$.
(2) $\mathrm{supp}(X) = \mathrm{Im}(w)$ and, for every $t \in \mathrm{supp}(X)$, $\mathrm{card}(X_t) = |w|_t$.

On the other hand, for the index set $\Lambda = \bigcup_{Y\in\mathrm{Sub}(A)}(\{Y\} \times \mathrm{supp}(Y))$ and the Λ-indexed family $(Y_s)_{(Y,s)\in\Lambda}$ whose (Y,s)-th coordinate is Y_s, precisely the s-th coordinate of the S-sorted set Y of the index $(Y,s) \in \Lambda$, let f be a choice function for $(Y_s)_{(Y,s)\in\Lambda}$, i.e., an element of $\prod_{(Y,s)\in\Lambda} Y_s$. Moreover, for every $w \in S^{\star}$ and $a \in \prod_{i\in|w|} A_{w(i)}$, let $M^{w,a} = (M^{w,a}_s)_{s\in S}$ be the finite S-sorted subset of A defined as $M^{w,a}_s = \{a_i \mid i \in w^{-1}[s]\}$, for every $s \in S$.

Now, for $(w,s) \in S^{\star}\times S$ and $(X,b) \in \Sigma_{w,s}$, let $F_{X,b}$ be the many-sorted operation from $\prod_{i\in|w|} A_{w(i)}$ into A_s that to an $a \in \prod_{i\in|w|} A_{w(i)}$ assigns b, if $M^{w,a} = X$ and $f(J(M^{w,a}), s)$, otherwise.

We will prove that the many-sorted Σ-algebra $\underline{A} = (A,F)$ is such that $J = \mathrm{Sg}_{\underline{A}}$. But before that it is necessary to verify that the definition of the many-sorted operations is sound, i.e., that for every $(w,s) \in S^{\star} \times S$, $(X,b) \in \Sigma_{w,s}$ and $a \in \prod_{i\in|w|} A_{w(i)}$, $s \in \mathrm{supp}(J(M^{w,a}))$ and for this it is enough to prove that $\mathrm{supp}(M^{w,a}) = \mathrm{supp}(X)$, because, by hypothesis, J is uniform and, by definition, $b \in J(X)_s$.

If $t \in \mathrm{supp}(M^{w,a})$, then $M^{w,a}_t$ is nonempty, i.e., there exists an $i \in |w|$ such that $w(i) = t$. Therefore, because $(X,b) \in \Sigma_{w,s}$, we have that $0 < |w|_t = \mathrm{card}(X_t)$, hence $t \in \mathrm{supp}(X)$.

Reciprocally, if $t \in \mathrm{supp}(X)$, $|w|_t > 0$, and there is an $i \in |w|$ such that $w(i) = t$, hence $a_i \in A_t$, and from this we conclude that $M^{w,a}_t \neq \varnothing$, i.e., that $t \in \mathrm{supp}(M^{w,a})$. Therefore, $\mathrm{supp}(M^{w,a}) = \mathrm{supp}(X)$ and, by the uniformity of J, $\mathrm{supp}(J(M^{w,a})) = \mathrm{supp}(J(X))$. But, by definition, $b \in J(X)_s$, so $s \in \mathrm{supp}(J(M^{w,a}))$ and the definition is sound.

Now we prove that, for every $X \subseteq A$, $J(X) \subseteq \mathrm{Sg}_{\underline{A}}(X)$. Let X be an S-sorted subset of A, $s \in S$ and $b \in J(X)_s$. Then, because J is algebraic, $b \in J(Y)_s$, for some finite S-sorted subset Y of X. From such an Y we will define a word w_Y in S and an element a_Y of $\prod_{i\in|w_Y|} A_{w_Y(i)}$ such that

(1). $Y = M^{w_Y,a_Y}$,
(2). $(Y,b) \in \Sigma_{w_Y,s}$, i.e., $b \in J(Y)_s$ and, for all $t \in S$, $\mathrm{card}(Y_t) = |w_Y|_t$, and
(3). $a_Y \in \prod_{i\in|w_Y|} X_{w_Y(i)}$,

then, because $F_{Y,b}(a_Y) = b$, we will be entitled to assert that $b \in \mathrm{Sg}_{\underline{A}}(X)_s$.

But taking into account that Y is finite iff $\mathrm{supp}(Y)$ is finite and, for every $t \in \mathrm{supp}(Y)$, Y_t is finite, let $\{\, s_\alpha \mid \alpha \in m\,\}$ be an enumeration of $\mathrm{supp}(Y)$ and, for every $\alpha \in m$, let $\{\, y_{\alpha,i} \mid i \in p_\alpha\,\}$ be an enumeration of the nonempty s_α-th coordinate, Y_{s_α}, of Y. Then we define, on the one hand, the word w_Y as the mapping from $|w_Y| = \sum_{\alpha\in m} p_\alpha$ into S such that, for every $i \in |w_Y|$ and $\alpha \in m$, $w_Y(i) = s_\alpha$ iff $\sum_{\beta\in\alpha} p_\beta \leq i \leq \sum_{\beta\in\alpha+1} p_\beta - 1$ and, on the other hand, the element a_Y of $\prod_{i\in|w_Y|} A_{w_Y(i)}$ as the mapping from $|w_Y|$ into $\bigcup_{i\in|w_Y|} A_{w_Y(i)}$ such that, for every $i \in |w_Y|$ and $\alpha \in m$, $a_Y(i) = y_{\alpha, i-\sum_{\beta\in\alpha} p_\beta}$ iff $\sum_{\beta\in\alpha} p_\beta \leq i \leq \sum_{\beta\in\alpha+1} p_\beta - 1$. From these definitions follow (1), (2) and (3) above. Let us observe that (1) is a particular case of the fact that the mapping M from $\bigcup_{w\in S^\star}(\{w\} \times \prod_{i\in|w|} A_{w(i)})$ into $\mathrm{Sub}_{\mathrm{f}}(A)$ that to a pair (w, a) assigns $M^{w,a}$ is surjective.

From the above and the definition of $F_{Y,b}$ we can affirm that $F_{Y,b}(a_Y) = b$, hence $b \in \mathrm{Sg}_{\underline{A}}(X)_s$. Therefore $J(X) \subseteq \mathrm{Sg}_{\underline{A}}(X)$.

Finally, we prove that, for every $X \subseteq A$, $\mathrm{Sg}_{\underline{A}}(X) \subseteq J(X)$. But for this, by the Proposition 2, it is enough to prove that, for every subset X of A, we have that $\mathrm{E}_{\underline{A}}(X) \subseteq J(X)$. Let $s \in S$ be and $c \in \mathrm{E}_{\underline{A}}(X)_s$. If $c \in X_s$, then $c \in J(X)_s$, because J is extensive. If $c \notin X_s$, then, by the definition of $\mathrm{E}_{\underline{A}}(X)$, there exists a word $w \in S^\star$, a many-sorted formal operation $(Y, b) \in \Sigma_{w,s}$ and an $a \in \prod_{i\in|w|} X_{w(i)}$ such that $F_{Y,b}(a) = c$. If $M^{w,a} = Y$, then $c = b$, hence $c \in J(Y)_s$, therefore, because $M^{w,a} \subseteq X$, $c \in J(X)_s$. If $M^{w,a} \neq Y$, then $F_{Y,b}(a) \in J(M^{w,a})_s$, but, because $M^{w,a} \subseteq X$ and J is isotone, $J(M^{w,a})$ is a subset of $J(X)$, hence $F_{Y,b}(a) \in J(X)_s$. Therefore $\mathrm{E}_{\underline{A}}(X) \subseteq J(X)$. □

From this last Theorem and the Proposition 3. we obtain

Corollary 1. *Let J be a many-sorted algebraic closure operator on an S- sorted set A. Then $J = \mathrm{Sg}_{\underline{A}}$ for some many-sorted Σ-algebra $\underline{A}$ iff J is uniform.*

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